An equation such as \( \sum_{n=1}^{\infty} a_n \) makes the series converge or diverge. In general statements, \( \sum_{n=1}^{\infty} a_n \) may stand for this.

A continuous function on a closed interval [a, b] is a function that is continuous at every point in the interval. The set of all such functions is denoted by \( C[a, b] \).

**Integral test & estimate**

- **Basic considerations**
  - If \( f(n) \) is integrable on [1, \( \infty \)) and \( f(n) \geq 0 \) for all \( n \geq 1 \), then the series \( \sum_{n=1}^{\infty} f(n) \) converges if and only if the improper integral \( \int_{1}^{\infty} f(x) \, dx \) converges.
  - The error in approximating the sum of the first \( n \) terms of the series by the integral is \( \int_{n}^{\infty} f(x) \, dx \).

- **Basic exercises**
  - Find the sum of the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).
  - Find the sum of the series \( \sum_{n=1}^{\infty} \frac{1}{n^3} \).

**Convergence tests**

- **Integral test & estimate**
  - If \( f(n) \) is increasing and \( f(n) \geq 0 \) for all \( n \geq 1 \), then the series \( \sum_{n=1}^{\infty} f(n) \) converges if and only if the improper integral \( \int_{1}^{\infty} f(x) \, dx \) converges.
  - The error in approximating the sum of the first \( n \) terms of the series by the integral is \( \int_{n}^{\infty} f(x) \, dx \).

- **Basic exercises**
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  - Find the sum of the series \( \sum_{n=1}^{\infty} \frac{1}{n^3} \).

**Power series**

- A power series in \( x \) is a sequence of polynomials in \( x \) of the form \( \sum_{n=0}^{\infty} a_n x^n \).
  - The power series is denoted \( \sum_{n=0}^{\infty} a_n x^n \).
  - Replacing \( x \) with a real number \( x \) a power series yields a number of real numbers. A power series converges at the points of the resulting series of real numbers.

- **Basic exercises**
  - Find the radius of convergence of the power series \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \).
  - Find the interval of convergence of the power series \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \).

**Convergence tests**

- **Basic**
  - **Direct comparison test**
    - Assume \( f(n) \) is a non-negative function, for which \( 0 \leq f(n) < g(n) \) for all \( n \) and for which \( \sum_{n=1}^{\infty} g(n) \) converges. Then \( \sum_{n=1}^{\infty} f(n) \) converges.
  - **Limit comparison test**
    - Assume \( f(n) \) and \( g(n) \) are non-negative functions for which \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = L \), where \( L \) is a finite non-negative number.
      - If \( L < 1 \), then \( \sum_{n=1}^{\infty} f(n) \) converges.
      - If \( L > 1 \), then \( \sum_{n=1}^{\infty} f(n) \) diverges.

- **Interpretations**
  - **Main**
    - **Area under a curve**
      - If \( y = f(x) \) is integrable on \( [a, b] \), then the area under the graph of \( f(x) \) from \( a \) to \( b \) is given by \( \int_{a}^{b} f(x) \, dx \).


**APPLICATIONS**

**GEOMETRY**

Areas of plane regions: Consider a plane region admitting a continuous function of the form \( f(x,y) \). The area of the region bounded by the graphs of \( f(x,y) = 0 \) and \( g(x,y) = 0 \) (where \( g \leq f \) on the interval \( [a,b] \) and \( f \) is continuous) is defined as the double integral

\[
\int_a^b \left( \int_{g(x)}^{f(x)} dy \right) dx
\]


**DIFFERENTIAL EQUATIONS**

**Examples**:

A differential equation (DE) was solved in the form of a solution to initial value problem, such as example of that type of boundary conditions.

1. The differential equation is the solution.
2. The general DE is the solution.
3. The particular DE is the solution.

The solution to the associated DE is known to be the general solution.

**Error bounds**

As an example, the remainder generally becomes smaller, and a guess Taylor polynomial provides a good approximation to the function value near the point of expansion.

**Approximations**

• Taylor's formula
• Error bounds
• Linear approximation

**NUMERICAL INTEGRATION**

• Simpson's rule
• Trapezoid rule
• Midpoint rule

**SEQUENCES & SERIES**

• Sequences are functions whose domain consists of all integers greater than or equal to some integer, usually 0. The integer is a sequence at or within a bounded, open or closed interval, or (ii) a sequence at or within a non-integer value.

• Elementary sequences: An arithmetic sequence, \( a_n = a_1 + (n-1)d \), is a sequence of terms that differ by a constant. It is a special case of a linear function, and the common difference is the slope of the line.

• Geometric sequence: A geometric sequence, \( a_n = a_1 r^{n-1} \), is a sequence of terms that differ by a constant. It is a special case of a linear function, and the common difference is the slope of the line.

• Series: A series is the sum of the terms of a sequence. It is a special case of a linear function, and the common difference is the slope of the line.

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Substitution. Refers to the Change of variable formula formalizes the Theory, and often the formula is used in reverse. For an integral to be replaced using the change of variables formula, the function \( f(x,y) = 0 \) and \( g(x,y) = 0 \) (where \( g \leq f \) on the interval \( [a,b] \) and \( f \) is continuous) is defined as the double integral

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**INTEGRATION FORMULAS**

• Indefinite integrals: Each function gives just one antiderivative (all others differ by a constant from that given).

• Definite integrals: Any integral function must be written as a polynomial plus a proper functional rational function. (degree of numerator less than degree of denominator) proper rational function with real coefficients has a partial fraction decomposition. It can be written as a sum of such terms each summed being either a constant over a power of a linear polynomial or a linear polynomial over a power of a linear polynomial. The form of the denominator and the degree of the numerator imply there could be sums over fixed integers.

• Further indefinite integrals: The following substitutions hold:

\[
\begin{align*}
\int \sin \theta d\theta &= -\cos \theta + C \\
\int \cos \theta d\theta &= \sin \theta + C \\
\int \tan \theta d\theta &= -\ln |\sec \theta| + C \\
\int \cot \theta d\theta &= \ln |\sin \theta| + C
\end{align*}
\]

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\[
\int_a^b \left( \int_{g(x)}^{f(x)} dy \right) dx
\]
**INTEGRATION FORMULAS**

Most multivariable integrals. Each formula gives just one antiderivative (all others differ by a constant from that given). Notable exceptions: the line integral on an open curve is defined only on an open curve between two points, and the other is defined only on the interval of the two points.

**FURTHER INTEGRATION FORMULAS**

The following formulas hold for any functions $f, g$, and $h$:

- $\int f(x) \, dx + \int g(x) \, dx = \int (f(x) + g(x)) \, dx$
- $\int f(x) \, dx - \int g(x) \, dx = \int (f(x) - g(x)) \, dx$
- $\int f(x) \, g(x) \, dx = \int f(x) \, g'(x) \, dx + \int g(x) \, f'(x) \, dx$
- $\int f(x) \, g'(x) \, dx = \int f(x) \, g(x) \, dx - \int g(x) \, f'(x) \, dx$
- $\int f(x) \, g(x) \, dx = \int f(x) \, g'(x) \, dx - \int g'(x) \, f(x) \, dx$

**COMMON DEFINITE INTEGRALS**

- $\int_a^b x^k \, dx = \frac{1}{k+1} b^{k+1} - \frac{1}{k+1} a^{k+1}$, for $k \neq -1$
- $\int_a^b \frac{1}{x} \, dx = \ln |b| - \ln |a| = \ln \left| \frac{b}{a} \right|$, for $a > 0$ and $b > 0$
- $\int_a^b e^{x} \, dx = e^b - e^a$
- $\int_a^b \sin x \, dx = -\cos x \bigg|_a^b = \cos a - \cos b$
- $\int_a^b \cos x \, dx = \sin x \bigg|_a^b = \sin b - \sin a$
- $\int_a^b \tan x \, dx = \ln |\sec x| \bigg|_a^b = \ln |\sec b| - \ln |\sec a|$
- $\int_a^b \sec x \, dx = \ln |\tan x + \sec x| \bigg|_a^b = \ln |\tan b + \sec b| - \ln |\tan a + \sec a|$

**APPLICATIONS**

**GEOMETRY**

- **Acrs of plane regions**: Consider a plane region admitting a parameterization $\gamma(t) = (x(t), y(t))$, $a \leq t \leq b$. A curve $\gamma(t)$ is parametrized appropriately, and the arc length is the integral of the magnitude of the derivative $\gamma'(t)$ from $a$ to $b$, that is,$$L = \int_a^b |\gamma'(t)| \, dt$$

**APPROXIMATIONS**

**TAYLOR’S FORMULA**

**PHYSICS**

- **Motion in one dimension**: Suppose a variable displacement $s(t)$ is known and consider the acceleration $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$. The net work done by a force $F(t)$ from time $t = t_0$ to $t = t_1$ is the integral of the force times the displacement$W = \int_{t_0}^{t_1} F(t) \, ds$,

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- **Sequences**: A sequence is a function whose domain consists of all integers greater than or equal to some integer, usually 0. The $n$th term of a sequence is denoted by $a_n$, and the sequence is written as $\{a_n\}$. A sequence is said to converge to a limit $L$ if $\lim_{n \to \infty} a_n = L$.

**NUMERICAL INTEGRATION**

**SERIES OF REAL NUMBERS**

- **Series**: A series is a sequence obtained by adding the values of another sequence $s_n = a_1 + a_2 + \cdots + a_n$. The value of a series $\sum_{n=1}^{\infty} a_n$ is the limit of the sequence of partial sums $s_n$ if this limit exists. If the series converges, the sequence of partial sums $\{s_n\}$ converges to a limit $S$.

**APPROXIMATIONS**

- **Taylor polynomials**: The $n$th degree Taylor polynomial of $f(x)$ at $a$ is $P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$. This is also the average of the left sum and right sums of the graph. The mid point result: $E_n(x) = f(x) - P_n(x)$.

- **Midpoint rule**: This evaluates the Riemann sums on a regular partition of $[a, b]$ and the midpoint of each subinterval.

- **Simpson’s rule**: The weighted sum $\int_a^b f(x) \, dx \approx \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$. This is also the integral of the quadrat that interpolates the function at the midpoint. For a regular partition of $[a, b]$ into $2m$ points, the formula is:

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- **Substitution**: Refers to the Change of variable formulas. Given any function $f(x)$ and a variable $u = g(x)$, $f(x)$ can be evaluated by transforming $x$ to $u$.

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**DIVERGENT INTEGRALS**

- **Examples**: A divergent integral $\int_a^\infty f(x) \, dx$ is of the following forms:
  - $\int_a^\infty \frac{1}{x^p} \, dx$, for $p \leq 1$.
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  - $\int_a^\infty \cos x \, dx$.
  - $\int_a^\infty \tan x \, dx$.
  - $\int_a^\infty \sec x \, dx$.
  - $\int_0^\infty \frac{1}{x^p} \, dx$, for $p \leq 1$.

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  - $\int_a^\infty \tan x \, dx$.
  - $\int_a^\infty \sec x \, dx$.
  - $\int_0^\infty \frac{1}{x^p} \, dx$, for $p \leq 1$.
A continuous function on a set is one in which each element is the limit of a sequence of elements in the set. The properties of continuity are often used to establish the existence of limits.

A remainder is:

\[ \text{If } \lim_{n \to \infty} a_n = L, \text{ then } \lim_{n \to \infty} (a_n - L) = 0. \]

\[ \text{If } \lim_{n \to \infty} a_n = L, \text{ then } \lim_{n \to \infty} (a_n - a) = L - a. \]

A series is a sum of a sequence of numbers.

\[ \text{If } \lim_{n \to \infty} a_n = L, \text{ then } \lim_{n \to \infty} \sum_{n=1}^{\infty} a_n = L. \]

\[ \text{If } \lim_{n \to \infty} a_n = L, \text{ then } \lim_{n \to \infty} \sum_{n=1}^{\infty} a_n = L. \]

A power series is a series of the form:

\[ \sum_{n=0}^{\infty} c_n (x-a)^n. \]

A geometric power series is one of the form:

\[ \sum_{n=0}^{\infty} c_n (x-a)^n. \]

\[ \sum_{n=0}^{\infty} c_n (x-a)^n = \frac{c_0}{1-x-a}. \]

A geometric series is a series of the form:

\[ \sum_{n=0}^{\infty} r^n. \]

\[ \text{If } |r| < 1, \text{ then } \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}. \]

\[ \text{If } |r| > 1, \text{ then } \sum_{n=0}^{\infty} r^n \text{ diverges.} \]

\[ \text{If } |r| = 1, \text{ then } \sum_{n=0}^{\infty} r^n \text{ is undefined.} \]

A power series converges for a value of x if and only if the radius of convergence is less than or equal to 1.

\[ \text{If } |x-a| < R, \text{ then the series converges.} \]

\[ \text{If } |x-a| > R, \text{ then the series diverges.} \]

\[ \text{If } |x-a| = R, \text{ then the series may converge or diverge.} \]

\[ \text{Generally, the sum of a power series is equal to the function it represents.} \]

\[ \text{If } f(x) \text{ is a function at } x = a, \text{ then } \sum_{n=0}^{\infty} f^{(n)}(a) (x-a)^n \text{ is the Taylor series expansion of } f(x) \text{ at } x = a. \]

\[ \text{The Taylor series expansion of } e^x \text{ at } x = 0 \text{ is:} \]

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \]

\[ \text{The Taylor series expansion of } \sin x \text{ at } x = 0 \text{ is:} \]

\[ \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}. \]

\[ \text{The Taylor series expansion of } \cos x \text{ at } x = 0 \text{ is:} \]

\[ \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}. \]

\[ \text{The Taylor series expansion of } \ln(1+x) \text{ at } x = 0 \text{ is:} \]

\[ \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}. \]

\[ \text{The Taylor series expansion of } (1+x)^a \text{ at } x = 0 \text{ is:} \]

\[ (1+x)^a = \sum_{n=0}^{\infty} \frac{a(a-1)(a-2)...(a-n+1) x^n}{n!}. \]

\[ \text{The Taylor series expansion of } e^{-x} \text{ at } x = 0 \text{ is:} \]

\[ e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}. \]

\[ \text{The Taylor series expansion of } \frac{1}{1-x} \text{ at } x = 0 \text{ is:} \]

\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n. \]

\[ \text{The Taylor series expansion of } \frac{1}{1+x} \text{ at } x = 0 \text{ is:} \]

\[ \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n. \]

\[ \text{The Taylor series expansion of } f(x) \text{ at } x = a \text{ is:} \]

\[ f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n. \]

\[ \text{The Taylor series expansion of } f(x) \text{ at } x = a \text{ is:} \]

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